Численные методы для задачи минимизации расходов при альтернативных ресурсах (Mean Field Game)

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Mean Field Games as the Nash Point at Differential Level

M.Y. Huang, P.E. Caines, and R. Malhamé implemented for population behavior

J.-M. Lasry and P.-L. Lions introduced standard terminology of MFG

A. Bensoussan, J. Frehse, P. Yam. Mean Field Games and Mean Field Type Control Theory. Springer, Berlin, 2013.

The Formulation of Direct (Hamilton-Jacobi-Bellman) Problem

$$\frac{\partial m}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 m}{\partial x^2} + \frac{\partial(\alpha m)}{\partial x} = 0 \quad \text{in } (0,T)\times(0,1)$$

$$m(0,x) = m_0(x) \quad \forall\, x \in (0,1)$$

$$\frac{\partial m}{\partial x}(t,0) = \frac{\partial m}{\partial x}(t,1) = 0 \quad \forall\, t \in (0,T)$$

The Cost Functional

$$J(m,\alpha) = \int_0^T \int_0^1 \left(\alpha^2 m/2 + g\right) \mathrm{d}x \, \mathrm{d}t$$

$$g(t,x,\tilde{m}) - g(t,x,m) \le \left(\tilde{m} - m\right) b(t,x,m)$$

The example

A. Lachapelle, J. Salomon, G. Turinici: Computational of mean field equilibria in economics. Mathematical Models and Methods in Applied Sciences, 2010, Vol. 20, P. 567-588

$$w(x,m)=\frac{c_0 x}{c_1+c_2 m}$$

$$f(t,x)=p(t)(1-c_3 x)$$

$$g(t,x,m):=\big(f(t,x)+w(x,m)\big)m$$

$$b(t,x,m)=\frac{\partial g}{\partial m}(t,x,m)=f(t,x)+w(x,m)+m\frac{\partial w}{\partial m}(x,m)$$

$$g(t,x,\tilde{m})-g(t,x,m)\le\big(\tilde{m}-m\big)b(t,x,m)$$

Differential minimization problem

$$\begin{cases} \inf\limits_{\alpha} \int_0^T \int_0^1 \left(\alpha^2 m/2 + g \right) \mathrm{d}x \, \mathrm{d}t, \\ \dfrac{\partial m}{\partial t} - \dfrac{\sigma^2}{2} \dfrac{\partial^2 m}{\partial x^2} + \dfrac{\partial(\alpha m)}{\partial x} = 0 \quad \text{ in } \ (0,T) \times (0,1) \end{cases}$$

$$\int_0^1 \left(v(T,x) m(T,x) - v(0,x) m_0(x) \right) \mathrm{d}x =$$

$$= \int_0^T \int_0^1 \left(\frac{\partial v}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 v}{\partial x^2} + \alpha \frac{\partial v}{\partial x} \right) m \, \mathrm{d}x \mathrm{d}t$$

$$\frac{\partial v}{\partial x}(t,0) = \frac{\partial v}{\partial x}(t,1) = 0 \quad \forall \, t \in (0,T)$$

The Lagrangian and the minimization problem

$$\mathfrak{I}(m,\alpha,v) := J(m,\alpha) + \int_0^T \int_0^1 \left(\frac{\partial v}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 v}{\partial x^2} + \alpha \frac{\partial v}{\partial x} \right) m \, \mathrm{d}x \, \mathrm{d}t -$$

$$- \int_0^1 \big(v(T,x)m(T,x) - v(0,x)m_0(x) \big) \, \mathrm{d}x$$

$$\inf_{(m,\alpha)} \sup_{v} \mathfrak{I}(m,\alpha,v)$$

The adjoint (Fokker-Planck) problem and an auxiliary equality

$$\frac{\partial v}{\partial t}+\frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2}+\alpha\frac{\partial v}{\partial x}+\frac{\alpha^2}{2}=-b(t,x,m)\quad\forall(t,x)\in[0,T]\times[0,1]$$

$$v(T,x)=0\quad\forall x\in(0,1)$$

$$\frac{\partial v}{\partial x}(t,0)=\frac{\partial v}{\partial x}(t,1)=0\quad\forall t\in(0,T)$$

$$\alpha=-\partial v/\partial x\quad\forall(t,x)\in[0,T]\times[0,1]$$

Solving the HJB problem

$$\frac{\partial m}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 m}{\partial x^2} + \frac{\partial(\alpha m)}{\partial x} = 0 \quad \text{in } (0,T)\times(0,1)$$

$$m(0,x) = m_0(x) \quad \forall\, x \in (0,1)$$

$$\frac{\partial m}{\partial x}(t,0) = \frac{\partial m}{\partial x}(t,1) = 0 \quad \forall\, t \in (0,T)$$

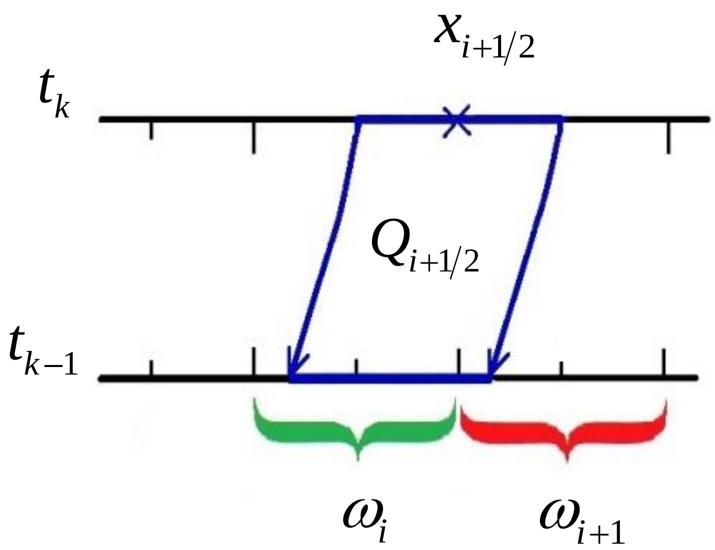
$$x_{i+1/2} = (i+1/2)h, \quad i = 0,\ldots,N-1, \;\; h = 1/N$$

$$t_k = k\tau, \quad k = 0,\ldots,M, \;\; \tau = 1/M$$

$$x_i = ih, \quad i = 0,\ldots,N.$$

Approximation of transfer part

$$\frac{\partial m}{\partial t} + \frac{\partial(\alpha m)}{\partial x} = f_1 \quad \text{with the right-hand side} \quad f_1 = \frac{\sigma^2}{2}\frac{\partial^2 m}{\partial x^2}$$



$$\hat{x}' = \alpha(t, \hat{x}), \quad t_k \geq t \geq t_{k-1}$$

$$\omega_i = [x_{i-1/2}, x_{i+1/2}] \ \forall i = 1, \ldots, N-1$$

$$m^h(t_k, x) = m^h_{k,i-1/2}(x_{i+1/2} - x)/h + m^h_{k,i+1/2}(x - x_{i-1/2})/h \ \ \forall x \in \omega_i$$

Approximation of transfer part

$$\int\limits_{Q_{i+1/2}} \left(\frac{\partial m}{\partial t} + \frac{\partial (\alpha m)}{\partial x} \right) \mathrm{d}t\, \mathrm{d}x = \int\limits_{Q_{i+1/2}} f_1 \, \mathrm{d}t\, \mathrm{d}x.$$

Gauss – Ostrogradskii theorem:

$$\int\limits_{Q_{i+1/2}} \left(\frac{\partial p}{\partial t} + \frac{\partial q}{\partial x} \right) \mathrm{d}t\, \mathrm{d}x = \int\limits_{S_{i+1/2}} \left(p n_t + q n_x \right) \mathrm{d}s$$

$$\int\limits_{Q_{i+1/2}} \left(\frac{\partial m}{\partial t} + \frac{\partial (\alpha m)}{\partial x} \right) \mathrm{d}t\, \mathrm{d}x = \int\limits_{x_i}^{x_{i+1}} m(t_k, x)\, \mathrm{d}x - \int\limits_{\hat{x}_i(t_{k-1})}^{\hat{x}_{i+1}(t_{k-1})} m(t_{k-1}, x)\, \mathrm{d}x$$

$$\frac{1}{\tau} m^h_{k,i+1/2} \approx \frac{1}{\tau h} \int\limits_{\hat{x}_i(t_{k-1})}^{\hat{x}_{i+1}(t_{k-1})} m^h(t_{k-1}, x)\, \mathrm{d}x + f_1(t_k, x_{i+1/2})$$

Approximation of diffusion part

$$-\frac{\sigma^2}{2}\frac{\partial^2 m}{\partial x^2} = f_2 \qquad \text{with the right-hand side} \quad f_2 = -\frac{\partial m}{\partial t} - \frac{\partial(\alpha m)}{\partial x}$$

$$\frac{\sigma^2}{2h^2}\left(-m^h_{k,i-1/2} + 2m^h_{k,i+1/2} - m^h_{k,i+3/2}\right) \approx f_2(t_k, x_{i+1/2})$$

$$-\frac{\sigma^2}{2h^2} m^h_{k,i-1/2} + \left(\frac{1}{\tau} + \frac{\sigma^2}{h^2}\right) m^h_{k,i+1/2} - \frac{\sigma^2}{2h^2} m^h_{k,i+3/2} =$$

$$= \frac{1}{\tau h} \int\limits_{\hat{x}_i(t_{k-1})}^{\hat{x}_{i+1}(t_{k-1})} m^h(t_{k-1}, x)\,\mathrm{d}x \quad \forall i = 0, \dots, N-1$$

Difference scheme for HJB problem

$$\hat{x}_i(t_{k-1}) = x_i - \tau\alpha^h_{k-1,i} \quad \text{and} \quad \hat{x}_{i+1}(t_{k-1}) = x_{i+1} - \tau\alpha^h_{k-1,i+1}$$

$$-\frac{\sigma^2}{2h^2} m^h_{k,i-1/2} + \left(\frac{1}{\tau} + \frac{\sigma^2}{h^2}\right) m^h_{k,i+1/2} - \frac{\sigma^2}{2h^2} m^h_{k,i+3/2} =$$

$$= \beta^{i+1/2}_{k-1,i-1/2} m^h_{k-1,i-1/2} + \beta^{i+1/2}_{k-1,i+1/2} m^h_{k-1,i+1/2} + \beta^{i+1/2}_{k-1,i+3/2} m^h_{k-1,i+3/2} \quad \forall\ i = 0,\dots,N-1$$

$$\beta^{i+1/2}_{k-1,i-1/2} = \frac{1}{8\tau}\left(1 + \frac{2\tau}{h}\alpha_{k-1,i}\right)^2$$

$$\beta^{i+1/2}_{k-1,i+1/2} = \frac{1}{8\tau}\left(3 - \frac{2\tau}{h}\alpha_{k-1,i}\right)\left(1 + \frac{2\tau}{h}\alpha_{k-1,i}\right) + \frac{1}{8\tau}\left(3 + \frac{2\tau}{h}\alpha_{k-1,i+1}\right)\left(1 - \frac{2\tau}{h}\alpha_{k-1,i+1}\right)$$

$$\beta^{i+1/2}_{k-1,i+3/2} = \frac{1}{8\tau}\left(1 - \frac{2\tau}{h}\alpha_{k-1,i+1}\right)^2$$

Simplified difference scheme for HJB problem

$$\hat{x}_i(t_{k-1}) = x_i - \tau\alpha^h_{k-1,i} \quad \text{and} \quad \hat{x}_{i+1}(t_{k-1}) = x_{i+1} - \tau\alpha^h_{k-1,i+1}$$

$$-\frac{\sigma^2}{2h^2} m^h_{k,i-1/2} + \left(\frac{1}{\tau} + \frac{\sigma^2}{h^2}\right) m^h_{k,i+1/2} - \frac{\sigma^2}{2h^2} m^h_{k,i+3/2} =$$

$$= \beta^{i+1/2}_{k-1,i-1/2} m^h_{k-1,i-1/2} + \beta^{i+1/2}_{k-1,i+1/2} m^h_{k-1,i+1/2} + \beta^{i+1/2}_{k-1,i+3/2} m^h_{k-1,i+3/2} \quad \forall\ i = 0,\dots,N-1$$

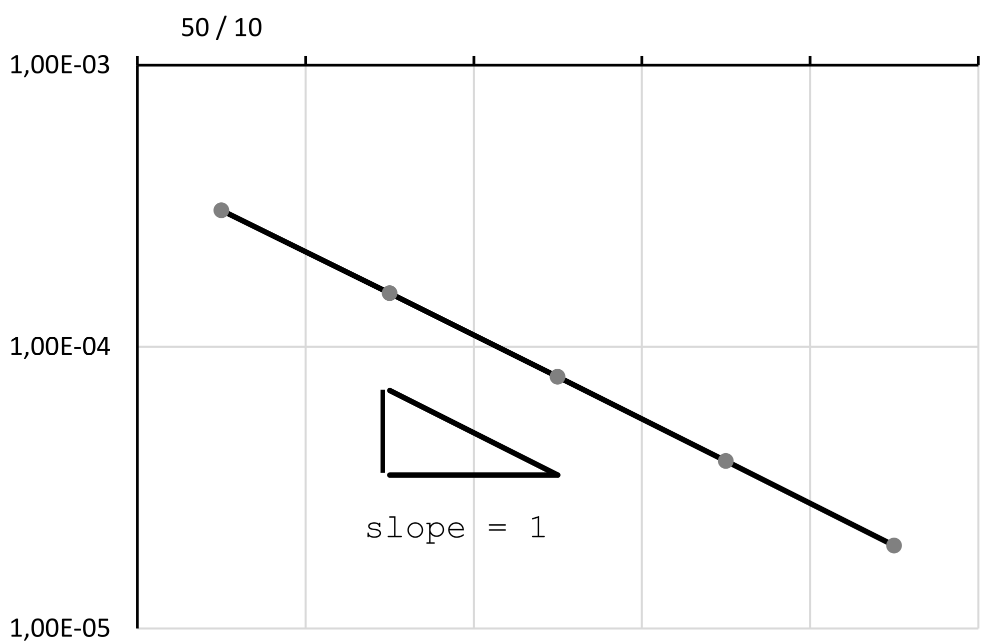
$$\beta^{i+1/2}_{k-1,i-1/2} = \frac{1}{8\tau}\left(1 + \frac{4\tau}{h}\alpha_{k-1,i}\right)$$

$$\beta^{i+1/2}_{k-1,i+1/2} = \frac{1}{8\tau}\left(3 + \frac{4\tau}{h}\alpha_{k-1,i}\right) + \frac{1}{8\tau}\left(3 - \frac{4\tau}{h}\alpha_{k-1,i+1}\right)$$

$$\beta^{i+1/2}_{k-1,i+3/2} = \frac{1}{8\tau}\left(1 - \frac{4\tau}{h}\alpha_{k-1,i+1}\right)$$

Convergence of HJB problem

N / M	Accuracy in L^1 -norm	Ratio	Order (log)
50 / 10	3.05 E-04		
100 / 20	1.55 E-04	1.97	0.98
200 / 40	7.82 E-05	1.98	0.99
400 / 80	3.92 E-05	1.99	0.99
800 / 160	1.97 E-05	2.00	1.00



$$O(h + \tau + h^2/\tau)$$

The discrete optimal problem

$$J^h(m^h,\alpha^h)=\sum_{k=0}^{M-1}\sum_{i=0}^{N-1}\left(r^h_{k,i+1/2}m^h_{k,i+1/2}+g^h_{k,i+1/2}\right)h\,\tau$$

$$r^h_{k,i+1/2}=(\alpha^h_{k,i})^2/4+(\alpha^h_{k,i+1})^2/4 \quad \text{and} \quad g^h_{k,i+1/2}=g(t_k,x_{i+1/2},m^h_{k,i+1/2})$$

$$\langle u,v\rangle=\sum_{i=0}^{N-1}u_{i+1/2}v_{i+1/2}h$$

$$J^h(m^h,\alpha^h)=\sum_{k=0}^{M-1}\left(\langle r^h_{k,\cdot},m^h_{k,\cdot}\rangle+\sum_{i=0}^{N-1}g^h_{k,i+1/2}\right)\tau$$

$$\mathbb{A}m^h_{\cdot,\cdot} = \begin{bmatrix} A & & & \\ -B_1 & A & & \\ & \ddots & \ddots & \\ & & -B_{M-1} & A \end{bmatrix} \begin{bmatrix} m^h_{1,\cdot} \\ m^h_{2,\cdot} \\ \vdots \\ m^h_{M,\cdot} \end{bmatrix} = \begin{bmatrix} B_0 m^h_{0,\cdot} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} = \mathbb{F} m^h_{0,\cdot}$$

$$A = \begin{bmatrix} \frac{1}{\tau} + \frac{\sigma^2}{2h^2} & -\frac{\sigma^2}{2h^2} & & \\ -\frac{\sigma^2}{2h^2} & \frac{1}{\tau} + \frac{\sigma^2}{h^2} & -\frac{\sigma^2}{2h^2} & \\ & \ddots & \ddots & \\ & & -\frac{\sigma^2}{2h^2} & \frac{1}{\tau} + \frac{\sigma^2}{2h^2} \end{bmatrix}$$

$$B_k = \begin{bmatrix} \beta^{1/2}_{k,1/2} & \beta^{1/2}_{k,3/2} & & \\ \beta^{3/2}_{k,1/2} & \beta^{3/2}_{k,3/2} & \beta^{3/2}_{k,5/2} & \\ & \ddots & \ddots & \\ & & \beta^{N-1/2}_{k,N-3/2} & \beta^{N-1/2}_{k,N-1/2} \end{bmatrix}$$

Discrete optimization problem and its Lagrangian

$$\begin{cases} \inf_{\alpha} J^h(m^h, \alpha^h), \\ \mathbb{A} m^h_{\cdot,\cdot} = \mathbb{F} m^h_{0,\cdot} \end{cases}$$

$$\sum_{k=1}^{M} \left(\left\langle A m^h_{k,\cdot}, v^h_{k,\cdot} \right\rangle - \left\langle B_{k-1} m^h_{k-1,\cdot}, v^h_{k,\cdot} \right\rangle \right) \tau =$$

$$= \left\langle m^h_{M,\cdot}, A v^h_{M,\cdot} \right\rangle - \left\langle m^h_{0,\cdot}, A v^h_{0,\cdot} \right\rangle + \sum_{k=0}^{M-1} \left(\left\langle m^h_{k,\cdot}, A v^h_{k,\cdot} \right\rangle - \left\langle m^h_{k,\cdot}, B^*_k v^h_{k+1,\cdot} \right\rangle \right) \tau$$

$$\mathfrak{I}^h(m^h, \alpha^h, v^h) := J^h(m^h, \alpha^h) +$$

$$+ \left\langle m^h_{M,\cdot}, A v^h_{M,\cdot} \right\rangle + \left\langle m^h_{0,\cdot}, A v^h_{0,\cdot} \right\rangle + \sum_{k=0}^{M-1} \left(\left\langle m^h_{k,\cdot}, A v^h_{k,\cdot} \right\rangle - \left\langle m^h_{k,\cdot}, B^*_k v^h_{k+1,\cdot} \right\rangle \right) \tau$$

$$\inf_{(m^h, \alpha^h)} \sup_{v^h} \mathfrak{I}^h(m^h, \alpha^h, v^h)$$

The FP problem

$$A v^h_{k,\cdot} = B^*_k v^h_{k+1,\cdot} + z^h_{k,\cdot} \quad \forall\, k = M-1, M-2, \ldots, 0$$

$$v^h_{M,\cdot} = \mathbf{0}$$

$$z^h_{k,i+1/2} = -b(t_k, x_{i+1/2}; m^h_{k,i+1/2}) - r^h_{k,i+1/2} \quad \forall\, k = 0, \ldots, M-1 \quad \forall\, i = 0, \ldots, N-1,$$

$$v^h_{k,-1/2} = v^h_{k,1/2}, \quad v^h_{k,N-1/2} = v^h_{k,N+1/2}.$$

$$\alpha^h_{\cdot,i} = -\left(v^h_{\cdot,i+1/2} - v^h_{\cdot,i-1/2}\right)\Big/ h \quad \forall\, i = 1, \ldots, N-1.$$

Discrete FP problem

$$\mathbb{B} v^h_{\cdot,\cdot} = \begin{bmatrix} A & -B_0^* & & \\ & A & -B_1^* & \\ & & \ddots & \\ & & & A \end{bmatrix} \begin{bmatrix} v^h_{0,\cdot} \\ v^h_{1,\cdot} \\ \vdots \\ v^h_{M-1,\cdot} \end{bmatrix} = \begin{bmatrix} z^h_{0,\cdot} \\ z^h_{1,\cdot} \\ \vdots \\ z^h_{M-1,\cdot} \end{bmatrix}$$

$$-\frac{\sigma^2}{2h^2} v^h_{k,i-1/2} + \left(\frac{1}{\tau} + \frac{\sigma^2}{h^2} \right) v^h_{k,i+1/2} - \frac{\sigma^2}{2h^2} v^h_{k,i+3/2} =$$

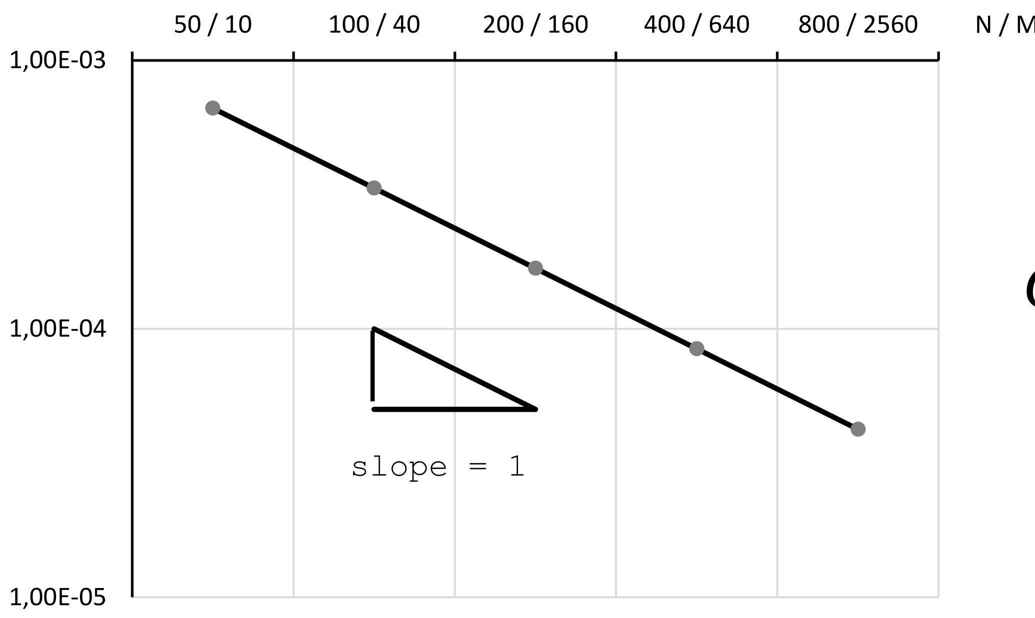
$$= \beta^{i-1/2}_{k,i+1/2} v^h_{k+1,i-1/2} + \beta^{i+1/2}_{k,i+1/2} v^h_{k+1,i+1/2} + \beta^{i+3/2}_{k,i+1/2} v^h_{k+1,i+3/2} + z^h_{k,i+1/2}$$

$$\forall\ k = M-1, M-2, \ldots, 0 \quad \forall\ i = 0, \ldots, N-1$$

$$v^h_{M,i+1/2} = 0 \quad \forall\, i = 0, \ldots, N-1$$

Convergence of FP problem

N / M	Accuracy in L_∞ - norm	Ratio	Order (log)
50 / 10	6,63E-04		
100 / 20	3,34E-04	1,98	0,99
200 / 40	1,68E-04	1,99	0,99
400 / 80	8,42E-05	1,99	1,00
800 / 160	4,22E-05	2,00	1,00



$$O(h + \tau + h^2/\tau)$$

Desintegration of estimate!

$$J^h(\tilde{m}^h, \tilde{\alpha}^h) - J^h(m^h, \alpha^h) \le \sum_{k=0}^{M-1} \sum_{i=0}^{N-1} \gamma_{k,i}^h \left(\left(\tilde{\alpha}_{k,i}^h \right)^2 - \left(\alpha_{k,i}^h \right)^2 \right) + \delta_{k,i}^h \left(\tilde{\alpha}_{k,i}^h - \alpha_{k,i}^h \right)$$

$$\gamma_{k,i}^h = \frac{\tau h}{4} \left(\tilde{m}_{k,i-1/2}^h + \tilde{m}_{k,i+1/2}^h \right)$$

$$\left(-\frac{\tau^2}{2h} \left(v_{k+1,i+1/2}^h - v_{k+1,i-1/2}^h \right) \left(\tilde{m}_{k,i+1/2}^h - \tilde{m}_{k,i-1/2}^h \right) \right)$$

$$\delta_{k,i}^h = \frac{\tau}{2} \left(\tilde{m}_{k,i+1/2}^h + \tilde{m}_{k,i-1/2}^h \right) \left(v_{k+1,i+1/2}^h - v_{k+1,i-1/2}^h \right)$$

The iterative algorithm

1. Solve discrete FP problem to get $v^h_{\cdot,\cdot}$

2. In cycle for $k = 0,\dots,M-1$

   2.1. compute $\tilde{\alpha}^h_{\cdot,i} = \left(v^h_{\cdot,i-1/2} - v^h_{\cdot,i+1/2}\right)/h;$

   2.2. compute $\tilde{m}^h_{k+1,\cdot}$ by discrete HJ equations;

3. If $J^h(m^h,\alpha^h) - J^h(\tilde{m}^h,\tilde{\alpha}^h) > Tol$ then go to 1;

4. Take $m^h_{\cdot,\cdot}$ and $\alpha^h_{\cdot,\cdot}$ as the approximate solution of the discrete optimization problem.

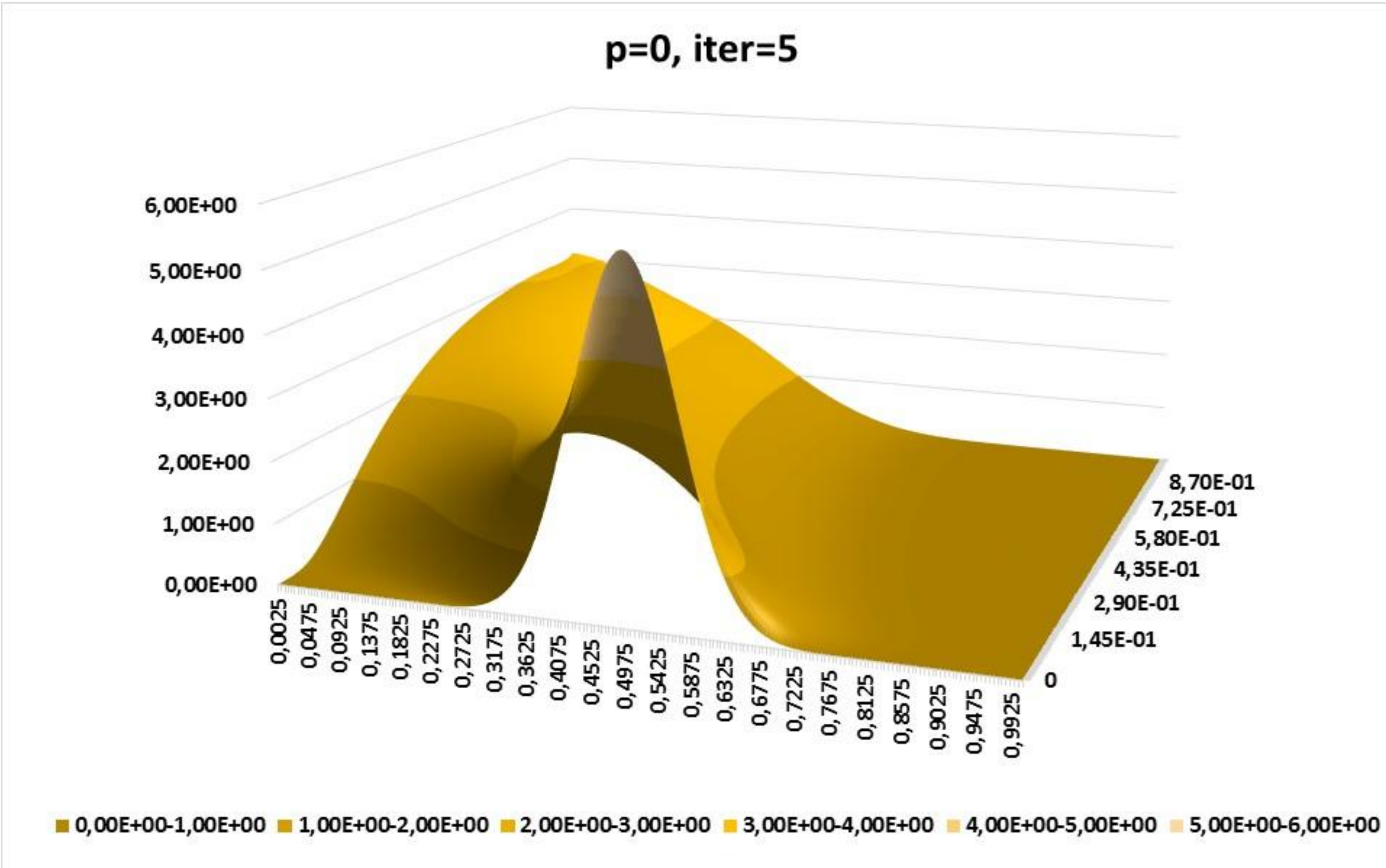
Testing of algorithms and methods

A. Lachapelle, J. Salomon, G. Turinici: Computational of mean field equilibria in economics. Mathematical Models and Methods in Applied Sciences, 2010, Vol. 20, P. 567-588.

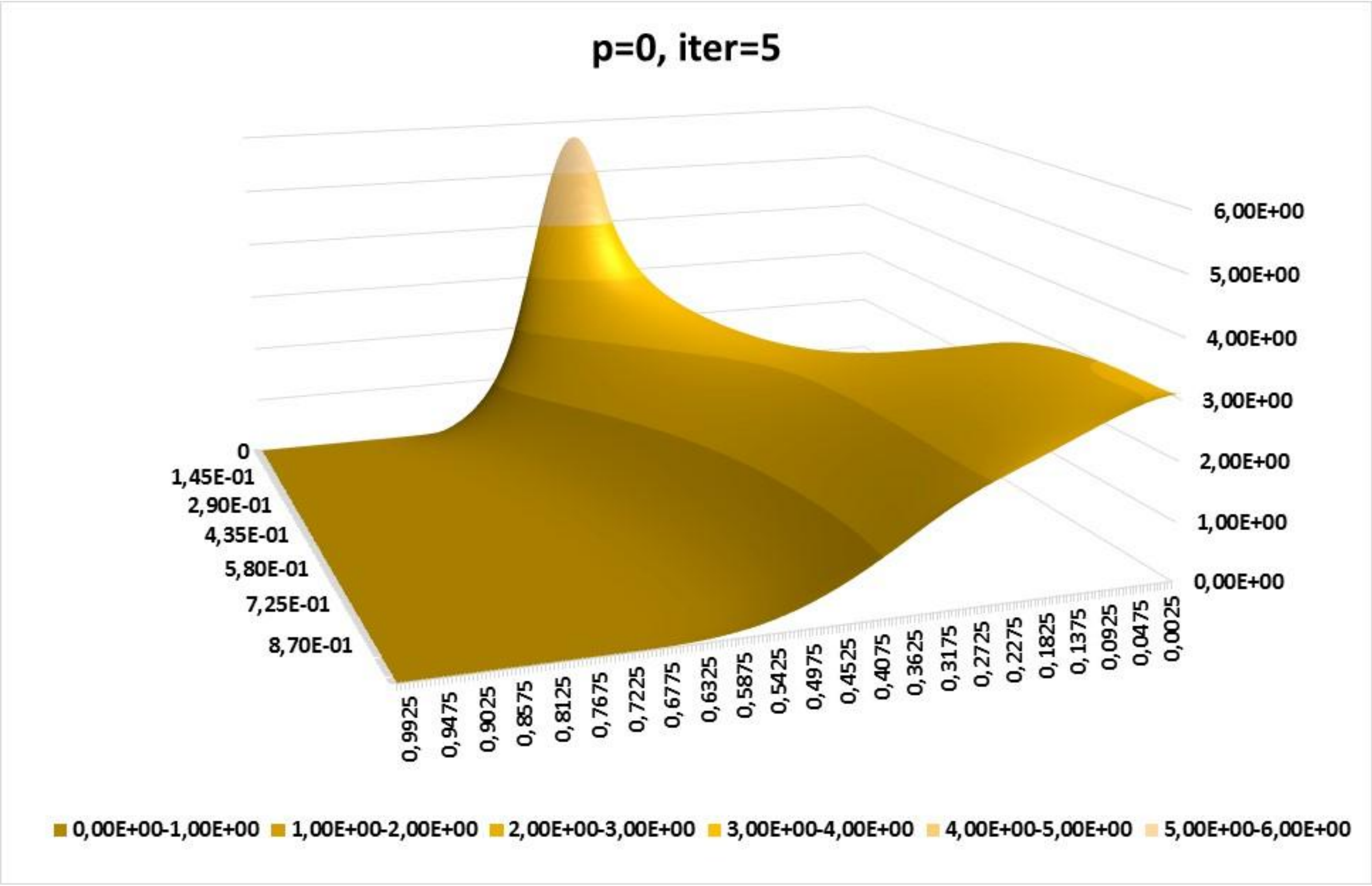
N of iteration	Criterion
1	6.73 E-02
2	3.61 E-02
3	1.84 E-02
4	2.51 E-02
5	0.00 E-02

1. Our schemes are monotone and converge with first order in space + time.
2. Both schemes have no switching.
3. The main iterative algorithm is simpler and has the better rate of convergence.
4. We have the real monotone criterion of convergence to the cost functional.

Price of electrical energy is 0



Price of electrical energy is 0



Thank you for attention!