Metrics of a fixed doubling dimension and efficient approximation of intractable combinatorial problems

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A vast majority of combinatorial optimization problems (e.g., Set Cover, Hitting Set Problem, Maximal Clique Problem, etc.) are known to be intractable and hardly approximable in general settings.

Meanwhile, for many actual special cases of these problems there are known efficient exact or approximation algorithms.

For instance, many combinatorial optimization problems become much more approximable being formulated in geometrical setting, in the Euclidean spaces of finite dimension.

But what can we say about their approximation in metric spaces?

In this lecture we will consider these aspects on Capacitated Vehicle Routing Problem (CVRP), the well-known generalization of the classic Traveling Salesman Problem.
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Traveling Salesman Problem (TSP)

**Problem statement**

**Input:** complete weighted graph $G = (V, E, w)$

**Required:** to find a Hamiltonian cycle of the minimum (or maximum) weight

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Complexity bounds

- exhaustive search \( \Theta((n - 1)!) \)
- dynamic programming \( \Theta(n^2 2^n) \)
Combinatorial optimization problems

- Combinatorial optimization problem $\mathcal{I}$

\[ I : OPT_I = \min \{ COST_I(x) : x \in X_I \} \]

$n := LEN(I)$ is instance length $I \in \mathcal{I}$.

- Algorithm is an arbitrary function $Alg : I \mapsto Alg(I) \in X_I$, computable in time $TIME_{Alg}(I)$

- Algorithm $Alg$ is called polynomial time, if

\[ TIME_{Alg}(I) = O(poly(LEN(I))) \ (I \in \mathcal{I}) \]

- Algorithm $Alg$ is called optimal, if

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is valid. Then $Alg$ is called $r$-approximation algorithm for the problem $\mathcal{I}$

- Algorithms with fixed accuracy bounds $r(I) = \text{const}$ and approximation schemes (PTAS or QPTAS) attract the most interest

- The problem $\mathcal{I}$ has PTAS (QPTAS), if for any $\varepsilon > 0$ there exists $(1 + \varepsilon)$-approximation polynomial time (quasi-polynomial time) algorithm $Alg_\varepsilon$

- For any PTAS, time complexity bound $TIME_{Alg_\varepsilon}(I) = O(poly(LEN(I)) = O(poly(n))$ depends on $n$ polynomially but can have an arbitrarily dependence on $\varepsilon$, e.g.

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Some known complexity and approximation results

### Complexity

- (Karp, 1972) TSP is strongly NP-hard
- (Sahni and Gonzales, 1976) TSP can not be approximated within $O(2^n)$ (unless $P = NP$)
- (Papadimitriou, 1977) Euclidean TSP is NP-hard
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**Approximation**

- (Christofides-Serdyukov, 1976) Metric TSP belongs to APX
- (Arora, 1996; Mitchell, 1996) First Polynomial Time Approximation Schemes (PTAS) for TSP on the plane
- (Arora, 1998) Euclidean TSP in $\mathbb{R}^d$ for any fixed $d > 1$ has EPTAS (but has no FPTAS unless $P = NP$)
- (Trevisan, 2000) TSP in the $d$-dimensional Euclidean space is APX-hard provided $d = \Omega(\log n)$
- (Talwar, 2004) QPTAS for the TSP in spaces of a fixed doubling dimension
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Contents

1. Metric spaces of a fixed doubling dimension

2. Snowflake embeddings and efficient approximation
Preliminaries

Definition

A metric space \((Z, \rho)\) has doubling dimension \(d\), if for an arbitrary \(z \in Z\) and \(R > 0\), there exist \(z_1, \ldots, z_t \in Z\), \(t \leq 2^d\) such that

\[
B(z, R) \subseteq \bigcup_{j=1}^{t} B(z_j, R/2).
\]

An example: metric \(l^d_\infty\) for \(d = 2\)

An example: metric \(l^d_2\)

Any \(d\)-dimensional Euclidean space is a metric space of doubling dimension \(O(d)\).
Lemma 1

Let $Z' \subset Z$ be a non-empty subspace of a finite aspect ratio $\Delta/\alpha$, where $\Delta = \sup\{\rho(u,v): u, v \in Z'\}$ and $\alpha = \inf\{\rho(u,v): \{u, v\} \subset Z'\}$. Then,

$$|Z'| \leq \left(\frac{2\Delta}{\alpha}\right)^d.$$
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Lemma 2

Let $(Z, \rho)$ be a metric space of doubling dimension $d > 1$, and $U \neq \emptyset$ be an arbitrary finite subspace of radius $R$. Then,

$$w(\text{MST}(U)) \leq 12R \cdot |U|^{1-1/d}.$$
Embeddings and snowflakes

Embedding to $l^D_p$

Finite metric space $(Z, \rho)$, $|Z| = n$ is embeddable to $l_p$ with distortion $L$, if there is a Lipschitz injective mapping $f : Z \to l_p$, s.t.

$$\rho(z_1, z_2) \leq \|f(z_1) - f(z_2)\| \leq L \cdot \rho(z_1, z_2)$$

for any $z_1, z_2 \in Z$.

Bourgain’s Theorem, 1985

For every $n$-point finite metric space there exists an embedding into $l_2$ with distortion $O(\log n)$.

Abraham, Bartal, and Neiman, 2011

Extended this result to the case of any $1 \leq p \leq \infty$ and dimension $O(\log n)$. 


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But ...

Logarithmic dimension is not good (recall Trevisan’s inapproximability result). We need embedding with fixed distortion into the Euclidean space of fixed dimension

Snowflakes

For any metric space $(Z, \rho)$ and $0 < \alpha < 1$, the space $(Z, \rho^\alpha)$ is called a snowflake.

Theorem (Assouad, Abraham et al.)

Let $(Z, \rho)$ be a metric space of doubling dimension $d$, $p > 1$, $0 < \alpha < 1$, $\theta > 0$ and $2^{192/\theta} \leq k \leq d$. There exists an embedding $f : Z \to l_p^D$, s.t.

$$L = O \left( k^{1+\theta} 2^{d/(pk)} / (1 - \alpha) \right) \quad \text{and} \quad D = O \left( \frac{d^{2d/k}}{\alpha \theta} \left( 1 - \frac{\log(1 - \alpha)}{\log k} \right) \right).$$

Example

Illustrate the applicability of this approach on approximation of the metric CVRP...
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**Example**

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Capacitated Vehicle Routing Problem

Input

A complete weighted graph $G = (Z, E, D, w)$, where $Z = X \cup \{y\}$,

- $X = \{x_1, \ldots, x_n\}$ are customers, $y$ is a depot
- $D: X \to \mathbb{Z}_+$ is a distribution of the customer demand
- $w: E \to \mathbb{R}_+$ specifies direct transportation costs

and an integer capacity bound $q \geq 3$. 
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Routes

An ordered pair $\mathcal{R} = (\pi, S_\mathcal{R})$, where
- $\pi = y, x_{i_1}, \ldots, x_{i_t}, y$ is a cycle in the graph $G$
- $S_\mathcal{R}: X \to \mathbb{Z}_+$ a distribution of the covered customer demand

is called a route of the cost

$$w(\mathcal{R}) = w(y, x_{i_1}) + w(x_{i_1}, x_{i_2}) + \cdots + w(x_{i_{t-1}}, x_{i_t}) + w(x_{i_t}, y).$$
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Feasible routes

The route $\mathcal{R} = (\pi, S_{\mathcal{R}})$ is called feasible, if

$$S_{\mathcal{R}}(x) = \begin{cases} \leq D(x), & x \in \{x_{i_1}, \ldots, x_{i_t}\} \\ = 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \sum_{x \in X} S_{\mathcal{R}}(x) \leq q.$$

Goal

To construct a set of feasible routes $\mathcal{S} = \sum_{\mathcal{R} \in S} w(\mathcal{R})$ of the minimum transportation cost that service the total customer demand.
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A complete weighted graph $G = (Z, E, D, w)$, where $Z = X \cup \{y\}$,
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The problem
$$w(\mathcal{S}) \equiv \sum_{\mathcal{R} \in \mathcal{S}} w(\mathcal{R}) \rightarrow \min$$
$$\sum_{\mathcal{R} \in \mathcal{S}} S_{\mathcal{R}}(x) = D(x) \quad (x \in X).$$
## Capacitated Vehicle Routing Problem

### Input

A complete weighted graph $G = (Z, E, D, w)$, where $Z = X \cup \{y\}$,
- $X = \{x_1, \ldots, x_n\}$ are customers, $y$ is a depot
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### Goal

To construct a set of feasible routes $S = \sum_{R \in S} w(R)$ of the minimum transportation cost that service the total customer demand.

### Additional constraints

1. for some $d > 1$, the weighting function $w$ is a metric of **doubling dimension** $d$;
2. the problem is supposed to have an optimal solution, whose number of routes does not exceed polylog $n$. 
Motivation

The data from survey ”Road-based goods transportation: a survey of real-world logistics applications from 2000 to 2015”

The spheres of applications
- Oil, gas and fuel transportation
- Retail applications
- Waste collection and management
- Mail and Small package delivery
- Food distribution
Consider the CVRP with $n = 5$ customers, $q = 2$ capacity and one time window

\[
S_{ITP} = \arg\min \{ w(S(x)) : x \in \{x_2, x_4, x_7, x_{11}, x_{13}\} \} 
\]
Haimovich - Rinnooy Kan approximation scheme

1. Relabel the customers in the order $r_1 \geq \ldots \geq r_{k-1} \geq r_k \geq \ldots \geq r_n$, where

   \[ r_i = w(y, x_i) \]

2. For the given $\varepsilon > 0$, find the minimum $\tilde{k} = \tilde{k}(\varepsilon, q, \beta)$, for which the relative error fulfils the inequality

   \[ e(\tilde{k}) = \frac{\text{CVRP}^*(X_{out}) + \text{ITP}(X_{in}) - \text{CVRP}^*(X)}{\text{CVRP}^*(X)} \leq \varepsilon. \]

   Here $X_{out} = \{x_1, \ldots, x_{\tilde{k}-1}\}$, $X_{in} = X \setminus X_{out}$, and $\text{ITP}(X_{in})$ is an upper cost bound for the partial solution obtained by ITP heuristic.

3. Find an exact solution $S_{DP}$ for the $X_{out}$ by dynamic programming.

4. Find an approximate solution $S_{ITP}$ for $X_{in}$.

5. Output the combined solution $S = S_{DP} \cup S_{ITP}$. 
Lemma 3

\[ \text{ITP}(X) = \omega(S_{ITP}) \leq 2 \left\lceil \frac{n}{q} \right\rceil \frac{\sum_{i=1}^{n} r_i}{n} + (1 - 1/q) \text{TSP}^*(X). \]

Lemma 4

For any metric CVRP,

\[ \text{CVRP}^*(X) \geq \max \left\{ \text{TSP}^*(X \cup \{y\}), 2r_1, \frac{2}{q} \sum_{i=1}^{n} r_i \right\}. \]

Lemma 5

For any partition \( X = X_{\text{out}} \cup X_{\text{in}} \) and any metric CVRP,

\[ \text{CVRP}^*(X_{\text{in}}) + \text{CVRP}^*(X_{\text{out}}) \leq \text{CVRP}^*(X) + 4(k - 1)r_k. \]
Lemma 6

Let \( X \subset B(0, R) = \{x \in l_2^D : \|x\|_2 \leq R\} \). Then,

\[
TSP^*(X) \leq C_D \cdot R + C_D^* \cdot R^{1/D} \left( \sum_{i=1}^{n} r_i \right)^{1-1/D},
\]

for \( C_D = D^2 \cdot 2^{D+1} \) and \( C_D^* = 4\sqrt{2} \cdot D^{(1+3/D)/2} \).

If the instance of the CVRP s.t.

1. \((Z, w)\), where \( Z = X \cup \{y\} \) is a metric space of doubling dimension \( d > 2 \)
2. for any \( x \in X \), \( a \leq w(x, y) \leq b \) for some \( 0 < a < b \), then

Lemma 7

\[
TSP^*_w(X') \leq TSP^*_{l_2^D}(X') \leq \frac{C_D L}{\sqrt{a}} \cdot R + \frac{C_D^* L}{\sqrt{a}} \cdot R^{1/D} \left( \sum_{x_i \in X'} r_i \right)^{1-1/D}.
\]
Haimovich - Rinnooy Kan scheme in metric space of a fixed doubling dimension

**Theorem**

A \((1 + \varepsilon)\)-approximate solution for an arbitrary instance CVRP satisfying conditions 1 and 2 can be obtained in time

\[
O\left(qk^32^k\right) + \text{TIME(TSP, } \beta, n) + O(n^2),
\]

where

\[
k = k(\varepsilon, q, \beta, D, a)
\]

\[
= O\left(\left(\frac{q}{\varepsilon}\right)^D \left(\frac{4\sqrt{2}\beta D^{(1+3/D)/2} \cdot L}{\sqrt{a}}\right)^D + \frac{2^D \cdot \beta L}{\sqrt{a}}\right) \cdot \left(\exp\left(\frac{q}{\varepsilon}\right)\right)^2,
\]

\text{TIME(TSP, } \beta, n) \text{ is running time of finding } \beta\text{-approximate solution of the auxiliary TSP instance, } D = O(d \cdot \log \log d) \text{ and } L = \text{poly } (d).
for some combinatorial problems including TSP, CVRP efficient approximation results (QPTAS or even PTAS) proposed initially for the finite dimensional Euclidean spaces can be extended to metric spaces of fixed doubling dimension.

there are two approaches for such an extension, based on embeddings and snowflakes and bypassing the embedding.

the question: ‘Can the QPTAS proposed by A.Das and C.Mathieu for the Euclidean CVRP be extended to metric spaces of a fixed doubling dimension without any restriction on the capacity growth?’ still remains open.
Thank you for your attention!